Computing Equilibria

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Google “christos”
### Games

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**Zero-sum game**

Min-max theorem von Neumann 1928: “a (probabilistic) equilibrium exists”
Non-zero sum games?

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Nash (1951):
“An equilibrium still exists”

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[First Parenthesis: Why study games?]

• Thought experiments that help us understand strategic rational behavior
• Studying them may help us understand the Internet
• Deep, rich subject
• It interacts exquisitely with computation]
How to compute a Nash equilibrium?

• All known algorithms are exponential
• By the way, it’s a combinatorial problem: one has to guess the supports
• Is it then NP-complete?
• No, because a solution always exists
…and why bother?
[a parenthesis

• Equilibrium concepts provide some of the most intriguing specimens of computational problems
• They are notions of *rationality*, aspiring models of *behavior*
• *Efficient computability is an important modeling prerequisite*
  
  “if your laptop can’t find it, then neither can the market…” ]

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Complexity of Nash

- Nash’s existence proof relies on Brouwer’s fixpoint theorem
- Finding a Brouwer fixpoint is a hard problem [HPV91]
- Not quite NP-complete, but as hard as any problem that always has an answer can be…
- But is Nash as hard? Or easier?
What is PPAD?

• Class of problems that always have a solution
• They all have the same existence proof

“If a finite directed graph has an unbalanced node, then it must have another one”
Exponential directed graph with indegree, outdegree < 2

Standard source (given)

(there must be a sink…)

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The four existence proofs

“if a directed graph has an unbalanced node, then it has another” PPAD
“if an undirected graph has an odd-degree node, then it has another” PPA
“every dag has a sink” PLS
“pigeonhole principle” PPP
Back to Nash

• For $n$ players with $s$ strategies each, input has length $ns^n$

• Exponential!
The embarrassing subject of many players [a parenthesis

• With games we are supposed to model markets, auctions, the Internet
• These have many players
• Thus they require exponential input!
The embarrassing subject of many players (cont.)

• These important games cannot require astronomically long descriptions
  “if your problem is important, then its input cannot be astronomically long…”

• Conclusion: Many interesting games are
  1. multi-player
  2. succinctly representable]
e.g., Graphical Games

- [Kearns et al. 2002] Players are vertices of a graph, each player is affected only by his/her neighbors
- If degrees are bounded by $d$, $ns^d$ numbers suffice to describe the game
- Also: multimatrix, congestion, location, anonymous, hypergraphical, …
An Easier Problem: Correlated equilibrium [Aumann 73]

Chicken:

- Two pure equilibria \{\text{me, you}\}
- Mixed \(\left(\frac{1}{2}, \frac{1}{2}\right)\) \(\left(\frac{1}{2}, \frac{1}{2}\right)\) payoff \(\frac{5}{2}\)
Correlated Equilibrium

• “Traffic signal” with payoff 3
  \[
  \begin{array}{cc}
  0 & \frac{1}{2} \\
  \frac{1}{2} & 0 \\
  \end{array}
  \]

• Compare with mixed Nash equilibrium
  \[
  \begin{array}{cc}
  \frac{1}{4} & \frac{1}{4} \\
  \frac{1}{4} & \frac{1}{4} \\
  \end{array}
  \]

• Even better with payoff 3 1/3
  \[
  \begin{array}{cc}
  \frac{1}{3} & \frac{1}{3} \\
  \frac{1}{3} & 0 \\
  \end{array}
  \]

Probabilities in a lottery drawn by an impartial outsider, and announced to each player separately

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Correlated equilibrium

• *Self-enforcing* distribution on the game states ("boxes")
• Generalizes the Nash equilibrium
  = uncorrelated (product) distribution
• Can be computed and optimized over by linear programming
Correlated equilibrium (cont)

- But how about the succinct case?
- *Can be computed in polynomial time by the “ellipsoid against hope” method* [Pa05]
- As long as the *utility expectation problem* can be solved in polynomial time
- Some cases of succinct games can even be optimized over [PR05]
Ellipsoid against hope: Details

- Existence $\equiv$ LP is unbounded
- LP has exponentially many variables
- “Solve” DLP by ellipsoid method
- When done (“no”), the facets used are also an infeasible RDLP, of polynomial size
- Solving their dual, RLP, solves LP
So…

• Nash equilibrium seems difficult (in PPAD, not known to be complete)
• Correlated equilibrium is easier
And suddenly, the summer of 2005

**Theorem [DGP05]:** Computing a Nash equilibrium is PPAD-complete even for 4 players

One key insight: *Games that do arithmetic!*
“Multiplication is the name of the game and each generation plays the same...”

Bobby Darren, 1961
The multiplication game

\[ z = x \cdot y \]

- If \( w \) plays 0, then it gets \( xy \).
- If \( w \) plays 1, then it gets \( z \), but \( z \) gets punished.

\( z \) wins when it plays 1 and \( w \) plays 0.
Reduction Brouwer ⇒ Nash: a very rough sketch

- Graphical games that do multiplication, addition, comparison, Boolean operations…
- Simulate the circuit that computes the Brouwer function by a huge graphical game
- “Brittle comparator” problem solved by averaging
- Simulate the graphical game by a 4-player game: 4-color the graph
Recall:

- Nash’s theorem reduces Nash to Brouwer
- This is a reduction in the opposite direction
So…. Brouwer $\equiv$ Nash
Later that Fall

[DP05, CD05]: 3-player Nash is also PPAD-complete

[Chen and Deng]: Even 2-player Nash!
game over?
Approximate Nash

Defecting players can only gain (additive) $\varepsilon$ (and utilities are normalized to $[0,1]$, 2 players)

- $\varepsilon = .75$ [KPS06]
- $\varepsilon = .5$ [DMP06], cf [FNS06]
- $\varepsilon = .3393...$ [ST07]
- Open: PTAS [any $\varepsilon > 0$ in time $O(n^{1/\varepsilon})$]
Anonymous games

• All players have the same set of strategies
• Each player has own utility
• But it depends on how many players [of each type] play each strategy
• Very Recent: PTAS [DP07] in the bounded strategy case (the only succinct one...)

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Anonymous games: The idea

- Each player’s strategy is a number in $[0,1]$
- Discretize to multiples of $\varepsilon$
- **Important Lemma:** payoffs change by $\sqrt{\varepsilon}$
- Search exhaustively for an approximate equilibrium
Finally, repeated games

(0,4) (1,1) (3,3) (4,0)

“threat point”

“individually rational region”

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Nash equilibria?

The Folk Theorem [ca. 1980]: Under very general conditions, any point in the IRR can be implemented as a Nash equilibrium.

Indeed [L2005]: For two players, a Nash equilibrium of the repeated game can be computed in polynomial time.
The Myth of the Folk Theorem

Theorem [BCIKPR2007]: For 3 or more players, the threat point is NP-hard to compute

Furthermore, finding a Nash equilibrium in a repeated game is PPAD-complete.
So…

- The Nash equilibrium is intractable
- Also in repeated games
- Approximation? Alternative notions? Special cases? Intractability of those?
- Computational results and concepts inform the game theoretic discourse
Thank You!