Computing Equilibria

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Games

	1/3	1/3	1/3
1/3	0, 0	1, -1	-1, 1
1/3	-1, 1	0, 0	1, -1
1/3	1, -1	-1, 1	0, 0

zero-sum game

Min-max theorem
von Neumann 1928:
“a (probabilistic)
equilibrium exists”

Non-zero sum games?

	1/4	1/4	1/2
1/4	**0, 0**	**2, -1**	**-1, 1**
1/4	**-2, 1**	**0, 0**	**1, -1**
1/2	**1, -1**	**-1, 1**	**0, 0**

Nash (1951):

"An equilibrium ***still*** exists"

[First Parenthesis: Why study games?

- Thought experiments that help us understand strategic rational behavior
- Studying them may help us understand the Internet
- Deep, rich subject
- It interacts exquisitely with computation]

How to compute a Nash equilibrium?

- All known algorithms are exponential
- By the way, it's a combinatorial problem: one has to guess the ***supports***
- Is it then NP-complete?
- ***No, because a solution always exists***

…and why bother?
[a parenthesis

- Equilibrium concepts provide some of the most intriguing specimens of computational problems
- They are notions of *rationality*, aspiring models of *behavior*
- ***Efficient computability is an important modeling prerequisite***

"if your laptop can't find it, then neither can the market…"]

Complexity of Nash

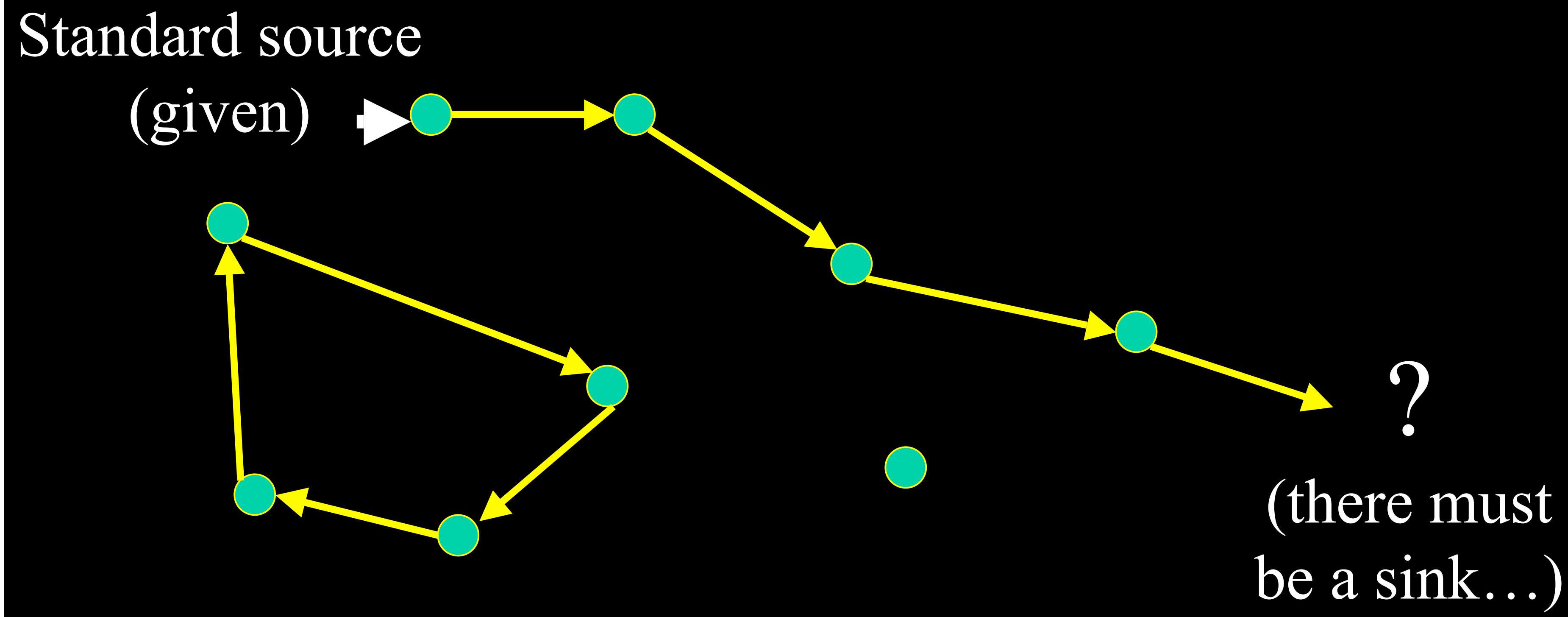
- Nash's existence proof relies on Brouwer's fixpoint theorem
- Finding a Brouwer fixpoint is a hard problem [HPV91]
- Not quite NP-complete, but as hard as any problem that always has an answer can be…
- Technical term: **PPAD-complete** [P 1991]
- But is Nash as hard? Or easier?

What is PPAD?

- Class of problems that always have a solution
- They all have the same existence proof

"If a finite directed graph has an unbalanced node, then it must have another one"

Exponential directed graph with indegree, outdegree < 2



The four existence proofs

"if a directed graph has an unbalanced node, then it has another" **PPAD**

"if an undirected graph has an odd-degree node, then it has another" **PPA**

"every dag has a sink" **PLS**

"pigeonhole principle" **PPP**

Back to Nash

- For n players with s strategies each, input has length

$$ns^n$$

- ***Exponenial!***

The embarrassing subject of many players [a parenthesis

- With games we are supposed to model markets, auctions, the Internet
- These have *many* players
- Thus they require exponential input!

The embarrassing subject of many players (cont.)

- These important games cannot require astronomically long descriptions

  "if your problem is important, then its input cannot be astronomically long…"

- Conclusion: Many interesting games are

1. multi-player
2. *succinctly representable*]

e.g., Graphical Games

- [Kearns *et al*. 2002] Players are vertices of a graph, each player is affected only by his/her neighbors
- If degrees are bounded by d, ns^d numbers suffice to describe the game
- Also: multimatrix, congestion, location, anonymous, hypergraphical, …

An Easier Problem: Correlated equilibrium [Aumann 73]

Chicken:

4,4	1,5
5,1	0,0

- Two pure equilibria {me, you}
- Mixed (½, ½) (½, ½) payoff 5/2

Correlated Equilibrium

- "Traffic signal"
  with payoff 3

- Compare with mixed
  Nash equilibrium

- Even better
  with payoff 3 1/3

0	½
½	0

1/4	1/4
1/4	1/4

1/3	1/3
1/3	0

Probabilities in a lottery drawn by an impartial outsider, and announced to each player separately

Correlated equilibrium

- ***Self-enforcing*** distribution on the game states ("boxes")
- Generalizes the Nash equilibrium
  = uncorrelated (product) distribution
- Can be computed and optimized over by linear programming

Correlated equilibrium (cont)

- But how about the succinct case?
- ***Can be computed in polynomial time by the "ellipsoid against hope" method*** [Pa05]
- As long as the ***utility expectation problem*** can be solved in polynomial time
- Some cases of succinct games can even be optimized over [PR05]

Ellipsoid against hope: Details

- Existence $\equiv$ LP is unbounded
- LP has exponentially many variables
- "Solve" DLP by ellipsoid method
- When done ("no"), the facets used are also an infeasible RDLP, of polynomial size
- Solving their dual, RLP, solves LP

So…

- Nash equilibrium seems difficult (in PPAD, not known to be complete)
- Correlated equilibrium is easier

And suddenly, the summer of 2005

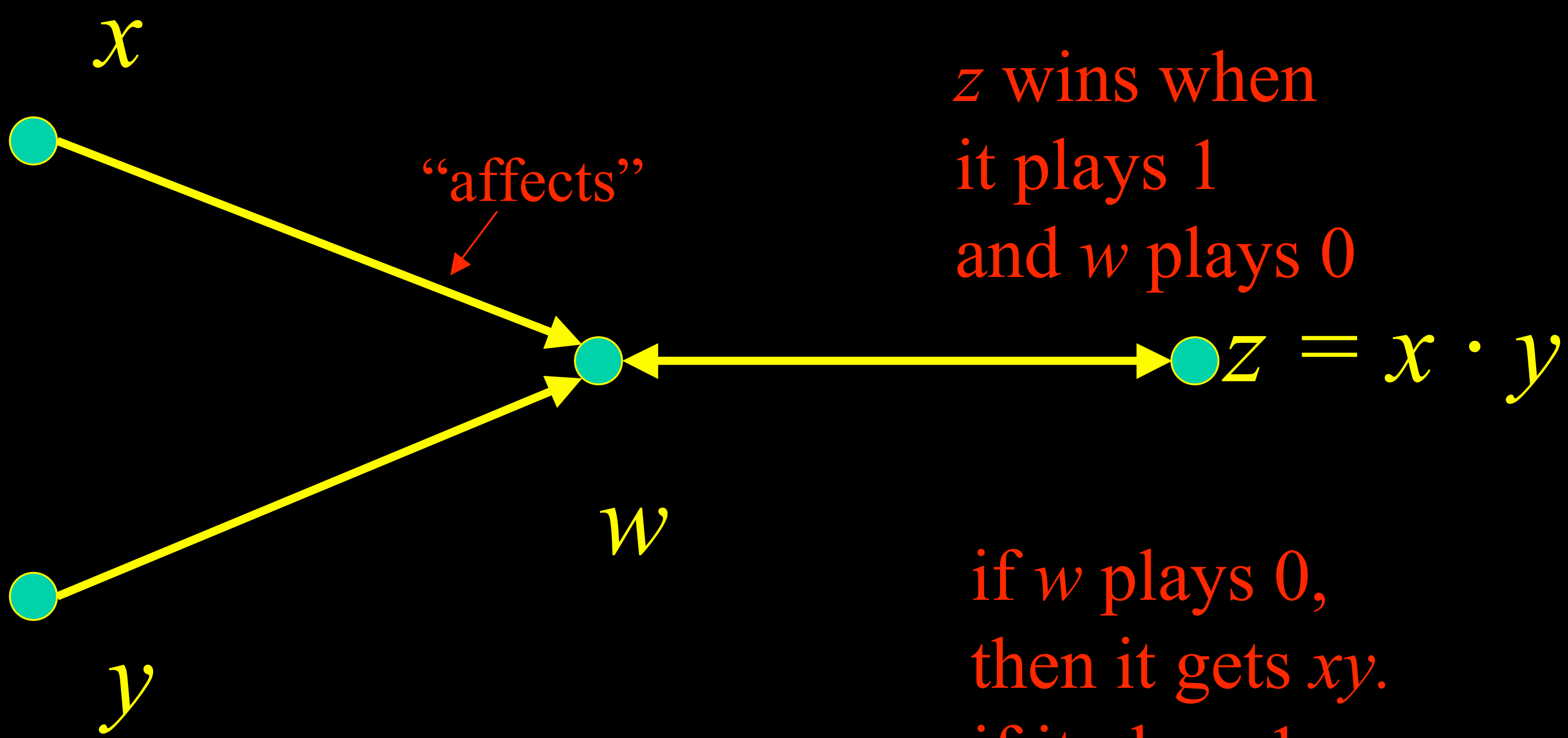
Theorem [DGP05]: Computing a Nash equilibrium is PPAD-complete even for 4 players

One key insight: ***Games that do arithmetic!***

“Multiplication
is the name of the game
and each generation
plays the same…”

Bobby Darren, 1961

The multiplication game



z wins when
it plays 1
and *w* plays 0

if *w* plays 0,
then it gets *xy*.
if it plays 1,
then it gets *z*,
but z gets punished

Reduction Brouwer $\Rightarrow$ Nash: a very rough sketch

- Graphical games that do multiplication, addition, comparison, Boolean operations…
- Simulate the circuit that computes the Brouwer function by a huge graphical game
- "Brittle comparator" problem solved by averaging
- Simulate the graphical game by a 4-player game: 4-color the graph

Recall:

- Nash’s theorem reduces Nash to Brouwer
- This is a reduction in the opposite direction

So….

Brouwer ≡ Nash

Later that Fall

[DP05, CD05]: 3-player Nash is also PPAD-complete

[Chen and Deng]: Even 2-player Nash!

game over?

Approximate Nash

Defecting players can only gain (additive) ε (and utilities are normalized to [0,1], 2 players)

- ε = .75 [KPS06]
- ε = .5 [DMP06], *cf* [FNS06]
- ε = .3393… [ST07]
- Open: PTAS [any $\varepsilon > 0$ in time $O(n^{1/\varepsilon})$]

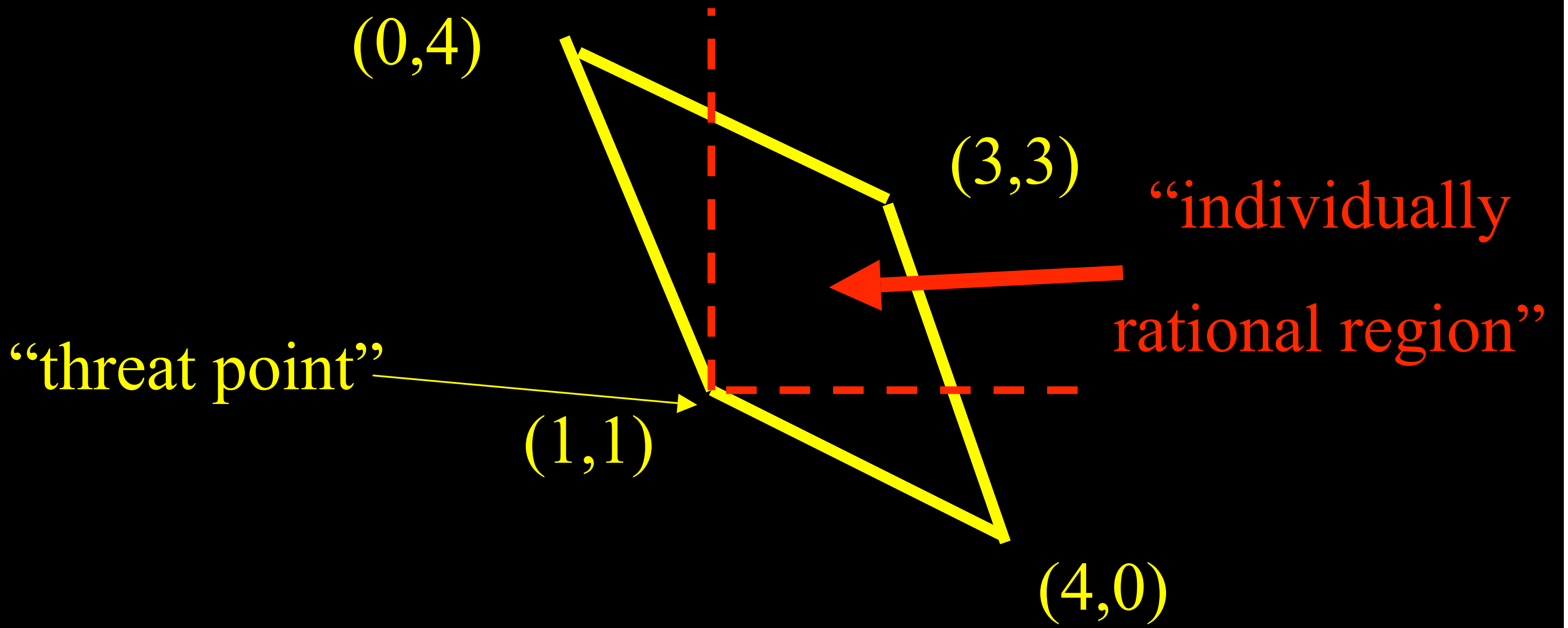
Anonymous games

- All players have the same set of strategies
- Each player has own utility
- But it depends on how many players [of each type] play each strategy
- *Very* Recent: PTAS [DP07] in the bounded strategy case (the only succinct one…)

Anonymous games: The idea

- Each player's strategy is a number in [0,1]
- Discretize to multiples of ε
- **Important Lemma:** payoffs change by √ε
- Search exhaustively for an approximate equilibrium

Finally, repeated games



Nash equilibria?

The Folk Theorem [ca. 1980]: Under very general conditions, any point in the IRR can be implemented as a Nash equilibrium.

Indeed [L2005]: For two players, a Nash equilibrium of the repeated game can be computed in polynomial time.

The Myth of the Folk Theorem

Theorem [BCIKPR2007]: For 3 or more players, the threat point is NP-hard to compute

Furthermore, finding a Nash equilibrium in a repeated game is PPAD-complete.

So…

- The Nash equilibrium is intractable
- Also in repeated games
- Aproximation? Alternative notions? Special cases? Intractability of those?
- Computational results and concepts inform the game theoretic discourse

Thank You!